LINEAR ALGEBRA — PRACTICE EXAM 2

(1) **Invertible matrices.**

Suppose A, B, and X are matrices that satisfy the relation $AX - A = B$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -1 & 2 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 3 & 4 \\ 1 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solve for X.

ANSWER: Solving for X symbolically, we see that

$$AX = B + A \quad \Rightarrow \quad X = A^{-1}(B + A) = A^{-1}B + I.$$

We not compute A^{-1}:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 3 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{bmatrix},$$

hence

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

Now solving for X:

$$\begin{aligned} X &= A^{-1}B + I \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 4 \\ 1 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 1 & 2 \\ -1 & 3 & 4 \\ 7 & 11 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 1 & 2 \\ -1 & 4 & 4 \\ 7 & 11 & 7 \end{bmatrix} \end{aligned}$$

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(2) **Column space and null space of a matrix.**

Let

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}.$$

Find a basis for $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

ANSWER: You can verify that

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence a basis for $\operatorname{Col}(A)$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}.$$

From the reduced matrix, we also see that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for $\operatorname{Nul}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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(3) **Change of basis.**

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the plane defined by $T(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} - 2\mathbf{x}$, where $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and let $\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$.

(a) Find the matrix A that satisfies $T(\mathbf{x}) = A\mathbf{x}$.
(b) Find the matrix P that satisfies $P[\mathbf{x}]_{\mathfrak{B}} = \mathbf{x}$.
(c) Find the matrix B that satisfies $[T(\mathbf{x})]_{\mathfrak{B}} = B[\mathbf{x}]_{\mathfrak{B}}$.
(d) Verify that $AP = PB$.

ANSWER:

(a) $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$, where $\{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$. We compute:

$$T(e_1) = \frac{3}{13}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -17 \\ 6 \end{bmatrix}$$
$$T(e_2) = \frac{2}{13}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 6 \\ -22 \end{bmatrix},$$

so $A = \dfrac{1}{13}\begin{bmatrix} -17 & 6 \\ 6 & -22 \end{bmatrix}$.

(b) $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$, where $\mathfrak{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Thus $P = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$.

(c) $B = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathfrak{B}} & [T(\mathbf{v}_2)]_{\mathfrak{B}} \end{bmatrix}$. We compute:

$$T(\mathbf{v}_1) = \frac{1}{13}\begin{bmatrix} -17 & 6 \\ 6 & -22 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{v}_1, \quad \text{so} \quad [T(\mathbf{v}_1)]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$T(\mathbf{v}_2) = \frac{1}{13}\begin{bmatrix}-17 & 6\\ 6 & -22\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}1\\-8\end{bmatrix} = \mathbf{v}_1 - 2\mathbf{v}_2, \quad \text{so} \quad [T(\mathbf{v}_2)]_{\mathfrak{B}} = \begin{bmatrix}1\\-2\end{bmatrix},$$

where $[T(\mathbf{v}_2)]_{\mathfrak{B}}$ is obtained by reducing

$$\begin{bmatrix}3 & 1 & 1\\ 2 & 5 & -8\end{bmatrix} \sim \begin{bmatrix}1 & -4 & 9\\ 2 & 5 & -8\end{bmatrix} \sim \begin{bmatrix}1 & -4 & 9\\ 0 & 13 & -26\end{bmatrix} \sim \begin{bmatrix}1 & -4 & 9\\ 0 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix}.$$

Thus $B = \begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}$.

(d)

$$AP = \frac{1}{13}\begin{bmatrix}-17 & 6\\ 6 & -22\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix} = \begin{bmatrix}3 & 1\\ 2 & -8\end{bmatrix}$$

$$PB = \begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix} = \begin{bmatrix}3 & 1\\ 2 & -8\end{bmatrix}.$$

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(4) **Subspaces.**

Let $V = \left\{\mathbf{x} \in \mathbb{R}^3 \colon \begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{x} = 0\right\}$. Show that V is a vector subspace of $\mathbb{R}^3$, or explain why it is not.

ANSWER: V is s subspace if

(i) $\vec{0}$ is in V,
(ii) V is closed under addition (if $\mathbf{a} \in V$ and $\mathbf{b} \in V$, then $\mathbf{a} + \mathbf{b} \in V$),
(iii) V is closed under scalar multiplication (if $\mathbf{a} \in V$, then $k\mathbf{a} \in V$ for all $k \in \mathbb{R}$).

We verify that V satisfies these three conditions. First $\vec{0} \in V$ since

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} = 0.$$

Second, if $\mathbf{a}$ and $\mathbf{b} \in V$ (that is, $\begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{a} = 0$ and $\begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{b} = 0$), then

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}(\mathbf{a}+\mathbf{b}) = \begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{a} + \begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{b} = 0,$$

so $\mathbf{a} + \mathbf{b} \in V$. Finally, if $\mathbf{a} \in V$ (that is, $\begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{a} = 0$), then

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}(k\mathbf{a}) = k(\begin{bmatrix}3 & 1 & 1\end{bmatrix}\mathbf{a}) = 0$$

so $k\mathbf{a} \in V$. Thus V is a vector subspace.